Distributivity (spectrum) of forcing notions

Marlene Koelbing and Wolfgang Wohofsky

joint work with Vera Fischer

Universität Wien (Kurt Gödel Research Center)

marlenekoelbing@web.de
wolfgang.wohofsky@gmx.at

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Definition

$\mathbb{P}$ is λ-distributive if it does not add a function $f : \lambda \to Ord$ with $f \notin V$.

$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

For maximal antichains A and B,

$$B \text{ refines } A :\Longleftrightarrow \forall q \in B\, \exists p \in A\, (q \leq p).$$

Proposition

$\mathbb{P}$ is λ-distributive if and only if for each family $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ of maximal antichains in $\mathbb{P}$, there exists a common refinement (i.e., a maximal antichain B such that B refines A_ξ for each $\xi < \lambda$).

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- $\omega \leq \mathfrak{h}(\mathbb{P}) \leq |\mathbb{P}|$
- $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$ ("the distributivity number")
  - $\omega_1 \leq \mathfrak{h} \leq \mathfrak{c}$ (since $\mathcal{P}(\omega)/\text{fin}$ is σ-closed and hence ω-distributive)
- Is there a generalization of $\mathfrak{h}$ to regular uncountable κ?
  - ...what about $\mathfrak{h}_\kappa := \mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa)$??
  - note that $\mathcal{P}(\kappa)/{<}\kappa$ is NOT σ-closed
  - in fact, $\mathcal{P}(\kappa)/{<}\kappa$ is NOT even ω-distributive; in other words: $\mathfrak{h}_\kappa = \omega$
  - so this definition is not interesting :-(
- The tower number $\mathfrak{t}$ has been generalized to κ:
  - $\mathcal{P}(\kappa)/{<}\kappa$ is not σ-closed, so straightforward generalization of $\mathfrak{t}$ yields ω
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Definition (Distributivity spectrum (with respect to fresh functions))

We say that $\lambda \in FRESH(\mathbb{P})$ if in some extension of V by $\mathbb{P}$,

there exists a fresh function on λ,

i.e., a function $f : \lambda \to Ord$ with

1. $f \notin V$, but
2. $f \restriction \gamma \in V$ for every $\gamma < \lambda$.

Note: $\lambda \in FRESH(\mathbb{P}) \iff \mathrm{cf}(\lambda) \in FRESH(\mathbb{P})$

So from now on, we only talk about regular cardinals λ.

Some basic facts:

- $\min(FRESH(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$
- $FRESH(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]$

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Let $\mathbb{C}_\mu$ be the forcing for adding μ many Cohen reals (μ arbitrarily large).

$$\mathit{FRESH}(\mathbb{C}_\mu) = ?$$

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$$FRESH(\mathbb{C}_\mu) = \{\omega\}$$

Theorem

If $\mathbb{P}$ satisfies $\mathbb{P} \times \mathbb{P}$ is δ-c.c. and $\lambda \geq \delta$, then $\lambda \notin FRESH(\mathbb{P})$.

Is $\mathbb{P}$ being δ-c.c. sufficient? No: consider a Suslin tree T (on ω_1)

- T is c.c.c. (i.e., ω_1-c.c.)
- BUT: $\omega_1 \in FRESH(T)$
- $\omega_2, \omega_3, \ldots \notin FRESH(T)$

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If $\mathbb{P}$ is δ-c.c. and $\lambda > \delta$, then $\lambda \notin FRESH(\mathbb{P})$.

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Lemma

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Recall: $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$

Theorem (Balcar-Pelant-Simon (Base Matrix Theorem))

$\mathcal{P}(\omega)/\text{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$.

Corollary

$\mathit{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$.

Theorem (Balcar-Simon; Shelah)

$\mathcal{P}(\kappa)/{<\kappa}$ collapses 2^κ to ω (assuming $2^{<\kappa} = \kappa$).

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$\mathcal{P}(\omega)/\text{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$.

Corollary

$\mathit{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$.

Theorem (Balcar-Simon; Shelah)

$\mathcal{P}(\kappa)/{<}\kappa$ collapses 2^κ to ω (assuming $2^{<\kappa} = \kappa$).

Corollary

$\mathit{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]$ (assuming $2^{<\kappa} = \kappa$).

Recall: $\mathfrak{h}(\mathbb{P}) = \min(\mathit{FRESH}(\mathbb{P}))$

Lemma

If $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \mathit{FRESH}(\mathbb{P})$.

Recall: $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$

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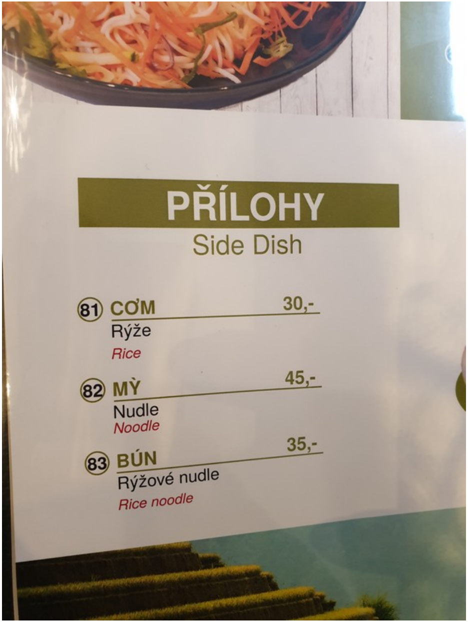
PŘÍLOHY
Side Dish
81 CƠM 30,-
Rýže
Rice
82 MỲ 45,-
Nudle
Noodle
83 BÚN 35,-
Rýžové nudle
Rice noodle

Definition

We say $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ is a distributivity matrix of height λ (for $\mathbb{P}$) if

- A_ξ is a maximal antichain in $\mathbb{P}$ (for each $\xi < \lambda$),
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  - A_η refines $A_\xi :\Longleftrightarrow \forall q \in A_\eta \, \exists p \in A_\xi \, (q \leq p)$
- the set $\{q \in \mathbb{P} : q \text{ intersects } \mathcal{A}\}$ is not dense in $\mathbb{P}$.
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Let $COM(\mathbb{P})$ denote the combinatorial distributivity spectrum of $\mathbb{P}$:

$$\lambda \in COM(\mathbb{P}) :\Longleftrightarrow \text{there exists a distributivity matrix of height } \lambda \text{ for } \mathbb{P}.$$

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$\min(COM(\mathbb{P})) = \min(FRESH(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

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$COM(\mathbb{P}) \subseteq FRESH(\mathbb{P})$

Proposition

$COM(\mathbb{P}) = FRESH(\mathbb{P})$ in case $\mathbb{P}$ is a complete Boolean Algebra

Recall:

- $FRESH(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$
- $FRESH(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]$ (assuming $2^{<\kappa} = \kappa$)

But note:

The Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is NOT complete!!

The same is true in the κ-case: $\mathcal{P}(\kappa)/{<}\kappa$ is NOT complete.

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Thank you for your attention and enjoy the Winter School...

Hejnice 2019

Distributivity (spectrum) of forcing notions

Marlene Koelbing and Wolfgang Wohofsky

joint work with Vera Fischer

Universität Wien (Kurt Gödel Research Center)

marlenekoelbing@web.de
wolfgang.wohofsky@gmx.at

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Hejnice, Czech Republic

27th Jan 2020

Observe that $\mathfrak{h} = \mathfrak{c}$ implies that

$$\{\mathfrak{h}\} = \mathit{FRESH}(\mathcal{P}(\omega)/\mathrm{fin}) = \mathit{COM}(\mathcal{P}(\omega)/\mathrm{fin}).$$

Theorem

It is consistent that $\mathfrak{h} < \mathfrak{c} = \omega_2$, and

$$[\mathfrak{h}, \mathfrak{c}] = \mathit{FRESH}(\mathcal{P}(\omega)/\mathrm{fin}) = \mathit{COM}(\mathcal{P}(\omega)/\mathrm{fin}) = \{\omega_1, \omega_2\}.$$

To prove that both ω_1 and ω_2 are in $\mathit{COM}(\mathcal{P}(\omega)/\mathrm{fin})$, we use two kinds of forcings: one adds a distributivity matrix of height ω_1, the other a distributivity matrix of height ω_2.

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Definition (The forcing for ω_2)

Let $T := \mathfrak{c}^{<\omega_2}$ and let $T^+ := \{\sigma \in T : |\sigma| \text{ is a successor}\}$. Define the forcing as follows: p is a condition if

- p is a finite function with $\mathrm{dom}(p) \subseteq T^+$,
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If G is a generic filter, let $a_\sigma := \bigcup_{p \in G} s^p_\sigma$, the matrix is $\{a_\sigma \mid \sigma \in T^+\}$.

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Problem

The rows of the generic matrix are not mad families (new reals are added).

Solution

Use iterated forcing to make sure that the rows are maximal in the end.

In each step of the iteration we force sets a_σ for all $\sigma \in \mathfrak{c}^{<\omega_2}$ for which they are not defined yet, and make sure that they are $\subseteq^*$ of the a_τ above (we get for free that they are almost disjoint to old sets in the same row). The iterands are defined as the forcing above, with the following changes:

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The forcing has the c.c.c.

Proof.

This is an easy Δ-system argument. □

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In the final model, the following holds for the generic matrix:

1. along branches through $\mathfrak{c}^{<\omega_2}$ we have $\subseteq^*$-decreasing sequences,
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This follows directly from the definition of the forcing, because the f's and h's ensure it. □

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Sketch of the proof.

Let $b \subseteq \omega$ infinite in the final model. Show that b is not a pseudo-intersection of any branch, and that b has infinite intersection with one element of each row.

Tower Assume σ is a branch through $\mathfrak{c}^{<\omega_2}$ and b is a pseudointersection of the sets along this branch. Use that all the information which is needed to decide something about b is bounded in $\mathfrak{c}^{<\omega_2}$, thus there exists some $\gamma < \omega_2$ such that the information at $\sigma\restriction\gamma$ is not relevant for b. So it is possible to decide that $m \in b$ and that $m \notin a_{\sigma\restriction\gamma}$ for arbitrarily large m. Thus b is not $\subseteq^*$ of $a_{\sigma\restriction\gamma}$. □

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An analogous forcing iteration of length ω_1 works to get a distributivity matrix of height ω_1.

To get a model in which $COM(\mathcal{P}(\omega)/\text{fin}) = \{\omega_1, \omega_2\}$ we start with a model of $\mathfrak{c} = \omega_2$ and combine the two forcings in the iteration; in the end, we have a distributivity matrix of height ω_1 and a distributivity matrix of height ω_2.

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Thank you for your attention and enjoy the Winter School...

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